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The stability of solutions of Vakhnenko's equation

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Abstract. Vakhnenko's equation has two families of travelling wave solutions. The method of Rowlands and Infeld is used to investigate whether these solutions are stable to long wavelength perturbations of small amplitude. The method predicts stability for both families of solutions. Some comments on the validity of the method are given.

1. Introduction

In a recent paper [1] Vakhnenko described the physical occurrence of the nonlinear evolution equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0 \tag{1.1}$$

(hereafter referred to as Vakhnenko's equation). He derived two families of travelling wave solutions of (1.1) corresponding to propagation in the positive and negative x directions respectively. It is of interest to investigate the stability of these solutions. A possible approach is to use the method devised by Rowlands and Infeld (see [2, chapter 3] and references therein). Their method is restricted to long wavelength perturbations of small amplitude. It has been applied successfully to a variety of generic nonlinear evolution equations (see [3] for example) and specific physical systems (see [4] for example). A particularly informative description of the method is given in [5] in the context of the Zakharov–Kuznetsov equation. Recently some criticism was levelled at the work in [5] by Das *et al* [6]; however, after a detailed reinvestigation of the problem, Das *et al* [6] vindicated the method used in [5].

The purpose of the present paper is to attempt to apply the Rowlands and Infeld method to Vakhnenko's equation. We shall find that the method appears to work successfully. However there are some doubts as to the validity of the analysis for the family of solutions that correspond to propagation in the positive x direction.

In section 2 we outline briefly the derivation of, and comment upon, the travelling wave solutions of (1.1). In section 3 we obtain the nonlinear dispersion relation for the perturbations to the solutions in section 2. In section 4 we examine the validity of the analysis in section 3 and present our conclusions.

2. Travelling wave solutions

In this section we recover Vakhnenko's travelling wave solutions of (1.1) and establish our notation, which is slightly different from that used by Vakhnenko. First we note that there are no stationary periodic solutions of (1.1) of the form $u = u(x)$. That being the case it is convenient to introduce a new dependent variable z and new independent variables η and τ defined by

$$z = (u - v)/|v| \qquad \eta = (x - vt)/|v|^{1/2} \qquad \tau = t|v|^{1/2}$$

where v is a non-zero constant. Then (1.1) becomes

$$z_{\eta\tau} + (zz_\eta)_\eta + z + c = 0 \tag{2.1}$$

where $c = \pm 1$ corresponding to $v \gtrless 0$. We now seek solutions of (2.1) of the form $z = z_0(\eta)$, so that z_0 satisfies

$$(z_0 z_{0\eta})_\eta + z_0 + c = 0. \tag{2.2}$$

After one integration (2.2) gives

$$\tfrac{1}{2}(z_0 z_{0\eta})^2 = \mathcal{F}(z_0) \tag{2.3}$$

where

$$\mathcal{F}(z_0) = -\tfrac{1}{3}z_0^3 - \tfrac{1}{2}cz_0^2 + \tfrac{1}{6}A \equiv -\tfrac{1}{3}(z_0 - z_1)(z_0 - z_2)(z_0 - z_3).$$

A is a constant and for periodic solutions z_1, z_2 and z_3 are real constants such that $z_1 \leqslant z_2 \leqslant z_0 \leqslant z_3$. On using results 236.00 and 236.01 of [7], we may integrate (2.3) to obtain

$$\eta = \frac{\sqrt{6}\,z_1}{\sqrt{z_3 - z_1}} F(\varphi, m) + \sqrt{6(z_3 - z_1)}\,E(\varphi, m) \tag{2.4}$$

where

$$\sin\varphi = \frac{z_3 - z_0}{z_3 - z_2} \qquad m = \frac{z_3 - z_2}{z_3 - z_1}. \tag{2.5}$$

$F(\varphi, m)$ and $E(\varphi, m)$ are incomplete elliptic integrals of the first and second kind respectively. We have chosen the constant of integration in (2.4) to be zero so that $z_0 = z_3$ at $\eta = 0$. The relations (2.4) and (2.5) (cf (7*a*) and (7*b*) of [1]) give the required solution in parametric form, with z_0 and η as functions of the parameter φ.

An alternative route to the solution is to follow the procedure described in [8]. We introduce a new independent variable ζ defined by

$$\frac{d\eta}{d\zeta} = z_0 \tag{2.6}$$

so that (2.3) becomes

$$\tfrac{1}{2}z_{0\zeta}^2 = \mathcal{F}(z_0). \tag{2.7}$$

By means of result 236.00 of [7], (2.7) may be integrated to give $p\zeta = F(\varphi, m)$, where $p^2 = (z_3 - z_1)/6$. Thus, on noting that $\sin\varphi = \mathrm{sn}(p\zeta|m)$, where sn is a Jacobian elliptic function, we have

$$z_0 = z_3 - (z_3 - z_2)\,\mathrm{sn}^2(p\zeta|m). \tag{2.8}$$

With result 310.02 of [7], (2.6) and (2.8) give

$$\eta = z_1\zeta + \sqrt{6(z_3 - z_1)}E(p\zeta) \tag{2.9}$$

where $E(p\zeta) \equiv E(\mathrm{am}\, p\zeta, m)$. Relations (2.8) and (2.9) are equivalent to (2.5) and (2.4) respectively and give the solution in parametric form with z_0 and η in terms of the parameter ζ.

We define the wavelength λ of the solution as the amount by which η increases when φ increases by 2π; from (2.4) we obtain

$$\lambda = \frac{2\sqrt{6}}{\sqrt{z_3 - z_1}}\left[z_1 K(m) + (z_3 - z_1)E(m)\right]$$

where $K(m)$ and $E(m)$ are complete elliptic integrals of the first and second kind respectively.

For $c = 1$ (i.e. $v > 0$), there are periodic solutions for $0 < A < 1$ with $\lambda < 0$, $z_2 \in (-1, 0)$ and $z_3 \in (0, 0.5)$. $A = 1$ gives the solitary wave limit

$$u = \tfrac{3}{2}v\,\mathrm{sech}^2(\zeta/2) \qquad \eta = -\zeta + 3\tanh(\zeta/2).$$

The periodic and solitary waves have a loop-like structure as illustrated in figure 1 of [1].

For $c = -1$ (i.e. $v < 0$), there are periodic waves for $-1 < A < 0$ with $\lambda > 0$, $z_2 \in (0, 1)$ and $z_3 \in (1, 1.5)$, but no solitary wave solutions. When $A = 0$ and $\lambda = 6$ the periodic wave solution simplifies to

$$u(\eta)/|v| = -\tfrac{1}{6}\eta^2 + \tfrac{1}{2} \qquad -3 \leqslant \eta \leqslant 3 \qquad u(\eta + 6) = u(\eta). \tag{2.10}$$

The periodic waves are illustrated in figure 2 of [1].

3. The nonlinear dispersion relation for the perturbations

We assume a perturbed solution of (2.1) in the form

$$z = z_0(\eta) + \mu\{\delta z(\eta)\exp[\mathrm{i}(k\eta - \omega\tau)] + \mathrm{CC}\} \tag{3.1}$$

where μ is a small positive constant characterizing the amplitude of the perturbation, $\delta z(\eta)$ is periodic with period λ, k is a real constant, ω is a constant (possibly complex), and CC denotes the complex conjugate of the preceding terms. Substitution of (3.1) into (2.1) and linearization with respect to μ yields

$$\mathcal{L}\delta z = f \tag{3.2}$$

where

$$\mathcal{L}\delta z = (z_0\delta z)_{\eta\eta} + \delta z$$

$$f = (-\omega k + k^2 z_0)\delta z + \mathrm{i}[\omega\delta z_\eta - 2k(z_0\delta z)_\eta].$$

As (2.2) implies that $\mathcal{L}z_{0\eta} = 0$, we may deduce that, for (3.2) to have periodic solutions, the condition $\langle z_0 z_{0\eta} f\rangle = 0$ must be satisfied; the averaging operation $\langle\cdot\rangle$ on an arbitrary function $h(\eta)$ is defined by

$$\langle h(\eta)\rangle = \mathrm{Fp}\,\frac{1}{\lambda}\int_0^{\lambda} h(\eta)\,\mathrm{d}\eta$$

where Fp denotes Hadamard's finite part (see [9, section 1.4]).

It is convenient to introduce the definitions

$$\beta_n = \left\langle \frac{z_0^{n-2}}{z_{0\eta}^2} \right\rangle \tag{3.3}$$

and

$$\alpha_n = \langle z_0^{n-1}\rangle. \tag{3.4}$$

Clearly $\alpha_1 = 1$ and, with (2.2), $\alpha_2 = -c$. Formally the solution of (3.2) is

$$\delta z = z_{0\eta}\phi \tag{3.5}$$

where

$$\phi_\eta = \left(D + \int z_0 z_{0\eta} f\,\mathrm{d}\eta\right) \Big/ (z_0 z_{0\eta})^2$$

and D is a constant. As δz appears on the right-hand side of (3.5) via f, we solve (3.5) iteratively by assuming that k is small in comparison with $2\pi/\lambda$ (so that the perturbations in (3.1) have long wavelength) and introduce the expansions

$$\delta z = \delta z_0 + k\delta z_1 + \cdots \qquad \omega = k\omega_1 + k^2\omega_2 + \cdots.$$

Thus f may be expressed in the form

$$f = kf_1 + k^2 f_2 + \cdots$$

where the f_n satisfy

$$\langle z_0 z_{0\eta} f_n\rangle = 0 \qquad n = 1, 2, \ldots. \tag{3.6}$$

At zeroth order (3.2) is $\mathcal{L}\delta z_0 = 0$ with solution $\delta z_0 = z_{0\eta}\phi_0$, where

$$\phi_{0\eta} = D_0/(z_0 z_{0\eta})^2 \tag{3.7}$$

and D_0 is a constant. Integration of (3.7) over a wavelength gives $D_0\beta_0 = 0$ and so $D_0 = 0$. It follows that

$$\delta z_0 = z_{0\eta}. \tag{3.8}$$

At first order (3.2) is $\mathcal{L}\delta z_1 = f_1$, where, with use of (3.8)

$$f_1 = \mathrm{i}[\omega_1 z_{0\eta\eta} - 2(z_0 z_{0\eta})_\eta].$$

On making use of (2.2) we find that (3.6) with $n = 1$ is satisfied identically and that (3.5) gives

$$\delta z_1 = z_{0\eta}\phi_1 \tag{3.9}$$

where

$$\phi_{1\eta} = [D_1 + \mathrm{i}\omega_1(z_0 z_{0\eta}^2 + \tfrac{1}{2}z_0^2 + cz_0) - \mathrm{i}(z_0 z_{0\eta})^2]/(z_0 z_{0\eta})^2 \tag{3.10}$$

and D_1 is a constant. Integration of (3.10) over a wavelength gives

$$D_1 = \mathrm{i}[1 - \omega_1(\alpha_0 + c\beta_1 + \tfrac{1}{2}\beta_2)]/\beta_0. \tag{3.11}$$

At second order (3.6) with $n = 2$ gives

$$\langle(z_0 z_{0\eta})^2\rangle - \omega_1\langle z_0 z_{0\eta}^2\rangle - \mathrm{i}\omega_1\langle(\tfrac{1}{2}z_0^2 + cz_0)\phi_{1\eta}\rangle - \mathrm{i}\langle(z_0 z_{0\eta})^2\phi_{1\eta}\rangle = 0 \tag{3.12}$$

where we have used (2.2) and (3.9). On substituting (3.10) into (3.12) we find that

$$\omega_1^2(\tfrac{1}{2}c + c^2\beta_2 + c\beta_3 + \tfrac{1}{4}\beta_4) - \mathrm{i}D_1[\omega_1(c\beta_1 + \tfrac{1}{2}\beta_2) + 1] = 0$$

with D_1 given by (3.11). Hence we obtain the main result of this section, namely the nonlinear dispersion relation for the perturbations

$$r_0 + r_1\omega_1 + r_2\omega_1^2 = 0 \tag{3.13}$$

where the coefficients r_0, r_1 and r_2 involve β_n ($n = 0, \ldots, 4$) and α_0. Now multiply (2.2) and (2.3) by $z_0^{n-2}/z_{0\eta}^2$ and integrate over a wavelength to obtain

$$n\alpha_{n-1} = -c\beta_n - \beta_{n+1} \tag{3.14}$$

and

$$\alpha_{n+1} = A\beta_n/3 - c\beta_{n+2} - 2\beta_{n+3}/3 \tag{3.15}$$

respectively; the five equations obtained from (3.14) with $n = 0, \ldots, 3$ and (3.15) with $n = 0$ are easily solved to give β_n ($n = 0, \ldots, 4$) in terms of α_0. Use of these results in (3.13) gives

$$r_0 = 1 \qquad r_1 = -\alpha_0 \qquad r_2 = \alpha_0^2/4 + c(1 + c\alpha_0)^2/4(A - c^3). \tag{3.16}$$

Finally, use of results 236.16, 310.00 and 310.02 of [7] in (3.4) gives

$$\alpha_0 = [z_1 + (z_3 - z_1)E(m)/K(m)]^{-1}.$$

(The β_n may be calculated directly from (3.3) by finite part integration; this provides a useful check on the above calculations.)

4. Conclusions

Let us consider the consequences of the dispersion relation (3.13). It has real roots if $r_1^2 \geqslant 4r_0r_2$. From (3.16) this condition is simply $c(c^3 - A) \geqslant 0$; this is satisfied for both of the families of travelling wave solutions discussed in section 2. Hence ω_1 is real and we conclude that both families of waves are stable to long wavelength perturbations. (The Rowlands and Infeld method makes no predictions for short wavelength perturbations.) The roots of the dispersion relation for $c = 1$ (i.e. $v > 0$) and $c = -1$ (i.e. $v < 0$) are displayed in figures 1 and 2, respectively.

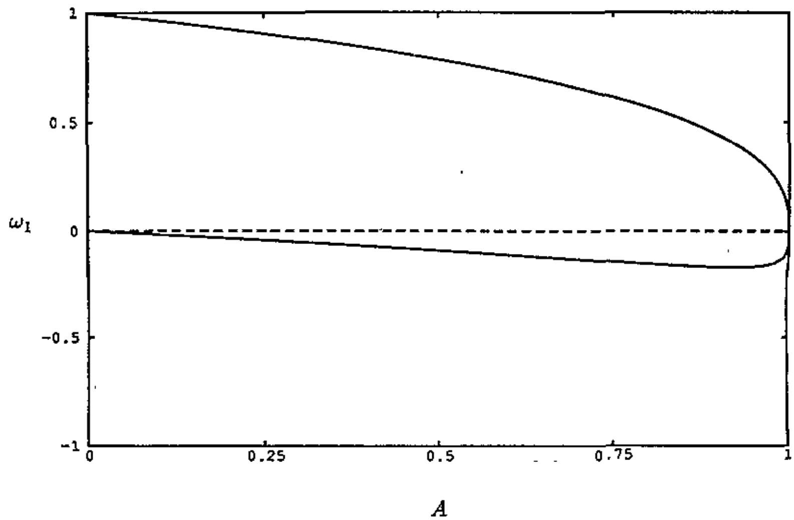

Figure 1. The roots of the nonlinear dispersion relation (3.13) for $c = 1$ (i.e. $v > 0$).

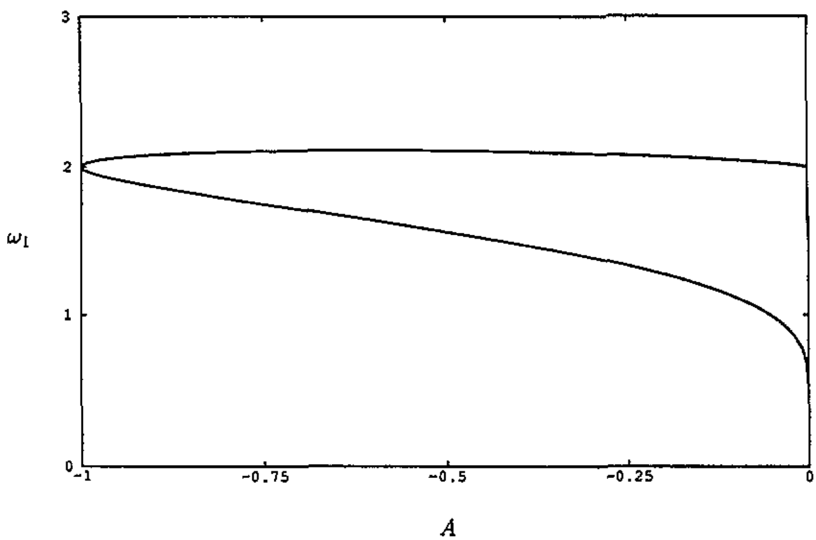

Figure 2. The roots of the nonlinear dispersion relation (3.13) for $c = -1$ (i.e. $v < 0$).

Before accepting this conclusion regarding stability, we review the assumptions and calculations leading to the dispersion relation (3.13).

For the family of travelling waves with $v > 0$ and $0 < A \leqslant 1$, the 'small' perturbation given by (3.1) with (3.8) and (3.9) becomes infinite at points on the loop-like solutions $z_0(\eta)$ where the slope is infinite. Thus, at these points, the usual linearization criterion, namely that $|\delta z|/|z|$ is bounded, fails and the derivation of (3.2) is invalid. However the Rowlands and Infeld method is based on average behaviour over a wavelength so that a more appropriate criterion would seem to be the boundedness of $\langle|\delta z|\rangle/\langle|z|\rangle$. This is satisfied and so, in this sense, the Rowlands and Infeld stability theory is valid.

For the family of periodic travelling waves with $v < 0$ and $-1 < A < 0$, there are no such problems and the Rowlands and Infeld method is certainly valid. For the periodic wave (2.10) for which $A = 0$, it can be seen that $z_{0\eta}$ is undefined at the points $\eta = 3 \pm 6n$, where n is an integer. In view of (3.8) and (3.9) we deduce that the perturbation given by (3.1) is also undefined at these points. However, it seems reasonable to assume that the wave (2.10) is stable since it is obtained straightforwardly by applying the limiting process $A \to 0$ (equivalent to $m \to 1$) to (2.8) and (2.9).

A useful check on the calculations leading to figure 2 is to follow the procedure described by Ziemkiewicz *et al* [10] as follows. Consider the weak linear limit in which $z_0 = 1$ and δz is constant in (3.1). In this case substitution of (3.1) into (2.1) and linearization gives the linear dispersion relation $\omega = k - k^{-1}$. The group velocity V_g for which $\omega = 0$, namely $V_g = 2$, should coincide with the linear limit of ω_1 from the fully nonlinear calculation, that is the value of ω_1 when $A = -1$. This is clearly the case in figure 2. Note that for $v > 0$ the weak linear limit is not a sinusoidal wave and so the procedure just described is not applicable to the curve in figure 1.

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